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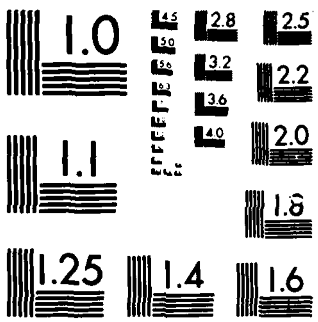

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All systems are subject to failure and must be repaired to be kept in working order. The word "repair" is used here in a broad sense. It can consist of replacement with a brand new item, or checkups at periodic intervals, or several other forms of "minimal" repairs. In this paper, we describe several kinds of "repairs" and maintenance models. After studying some properties of the stochastic process of failure times, ~~we~~ the authors compare different maintenance models by comparing the expected number of failures in time t. Keywords: life distributions; proportional hazards; mathematical models.

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Failure Times in Maintenance Models

by

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Abstract

All systems are subject to failure and must be repaired to be kept in working order. The word 'repair' is used here in a broad sense. It can consist of replacement with a brand new item, or checkups at periodic intervals, or several other forms of 'minimal' repairs. In this paper, we describe several kinds of 'repairs' and maintenance models. After studying some properties of the stochastic process of failure times, we compare different maintenance models by comparing the expected number of failures in time t.

1. Introduction.

All systems are subject to failure and must be repaired to be kept in working order. The word 'repair' is used here in a broad sense. It can consist of replacement with a brand new item, or checkups at periodic intervals, or several other forms of 'minimal' repairs. In this paper, we describe several kinds of 'repairs' and study some properties of the stochastic process of failure times under various maintenance models.

Consider a system consisting of only one unit. This unit is put into use at time $t = 0$ and has life distribution F. When it fails, we can perform a 'perfect' repair or a 'minimal' repair. 'Perfect' repair means that a new unit whose life distribution is F is put in the place of the failed unit. A 'G-minimal repair' means that the failed unit is replaced by a unit with life distribution G and age equals to the 'effective age' of the failed unit. More formally, let the failure time of the failed unit as computed from time zero or the last perfect repair, whichever came last, be y. Then the probability that the life of the G-minimal repaired replacement exceeds x is $\bar{G}(x+y)/\bar{G}(y)$ for $x \geq 0$. This definition of minimal repair was proposed by Ascher (1978) and has been used by Brown and Proschan (1983).

Brown and Proschan (1983) considered the following maintenance model. A system consisting of a single unit starts out with a unit whose life distribution is F. Whenever a failure occurs, a coin with probability p for heads is tossed independent of previous history. If the coin comes up heads a perfect repair is performed. Otherwise an F-minimal repair is performed. Notice that each epoch of perfect repair is a regeneration point for the process of failure times. Brown and Proschan (1983) obtained the distribution of the time between

perfect repairs (which is also the same as the waiting time for the first perfect repair) and established some of its monotonicity properties in terms of similar properties of F.

It is easy to see that the distribution of the time between perfect repairs depends in a simple fashion on the distribution of the number of minimal repairs before time t in another maintenance model called the Forever Minimal Repair (FMR) model where only G-minimal repairs are performed (see Theorem 2.1). This leads to an alternative derivation of Lemma 2.1 of Brown and Proschan (1983).

What are the interesting questions concerning maintenance models? By very definition, a system can be maintained indefinitely under all repair models, if the lifetimes of repaired units are unbounded to the right. Some models will be more expensive to maintain than others. In this paper we will study the failure time processes of some maintenance models and compare the expected number of failures in time t, which is roughly related to the cost of maintaining the system till time t.

A maintenance model where perfect repairs are performed at each failure is called the Remove and Replace (RAR) model. The failure times in this model are easily studied by using standard renewal theory. In Section 2 we compare the expected number of failures in the RAR and FMR models. It is shown that the RAR model has smaller expected number of failures if the unit life distribution is IFRA. On the other hand, the FMR model has a smaller expected number of failures if the unit life distribution is DFRA.

In Section 3 we consider other maintenance models where two types of minimal repairs are available. These minimal repairs may be chosen at random before the start of the process and fixed ever after (one-shot random repair model), or can be chosen at random at start and alternated thereafter (alternating repair

model) or can be chosen at random at start and again, independently, at each failure (completely random repair model). We then compare the expected number of failures in time t under all these maintenance models.

2. Maintenance models with only one type of minimal repair.

In this section we consider maintenance models with only one type of minimal repair. Of course, if we do not wish to use minimal repair, we can use perfect repair, which is assumed to be always available.

Consider the Remove and Replace (RAR) model in which a failed unit is replaced with a new unit; that is a perfect repair is performed at each failure. The lifetimes of all the units are independent and identically distributed with common distribution function F. Then the number of failures before t, $\overline{N}(t)$, is a standard renewal process. It is well known that the probability generating function $\phi(t, q)$ of $\overline{N}(t)$ is given by

$$\phi(t, q) = \overline{F}_1(t) + \sum_{n=1}^{\infty} q^n[\overline{F}_{n+1}(t) - \overline{F}_n(t)]$$

$$= (1 - q) \sum_{n=1}^{\infty} q^{n-1}\overline{F}_n(t),$$

where $\overline{F}_n$ denotes the n-fold convolution of F. Furthermore, if $\int_{-\infty}^{\infty}\overline{F}(t)dt = \mu < \infty$, then the renewal theorem states that

$$\frac{\overline{N}(t)}{t} \to \frac{1}{\mu} \quad \text{wp1} \tag{2.1}$$

and

$$E[\frac{\overline{N}(t)}{t}] \to \frac{1}{\mu}. \tag{2.2}$$

We refer to Barlow and Proschan (1975), p. 167 for the proof of these results. These well known results will be used as yardsticks for later comparisons.

In contrast to the above model consider the Forever Minimal Repair (FMR) model in which each failed unit is G-minimally repaired. More formally, if there is a failure at time t, then it is replaced with a working unit with life distribution G and age t, i.e., with a random life T whose distribution is given by

$$P(T > s) = \frac{\overline{G}(s+t)}{\overline{G}(t)}, \quad s \geq 0.$$

We also assume that the life distribution of the first unit is G. Let N(t) be the number of failures before t. Theorem 2.1 below obtains the probability generating function of N(t).

Theorem 2.1. The probability generating function of N(t) is given by

$$\phi(t, q) = \overline{G}^{p}(t) \quad \text{where } p = 1 - q. \tag{2.3}$$

Proof.

Let X_n be the time of the n^{th} failure and let F_n be the distribution of X_n. Then for $n \geq 1$,

$$\begin{aligned} P[N(t) = n] &= \overline{F}_{n+1}(t) - \overline{F}_n(t) &(2.4)\\ &= P(X_n < t \leq X_{n+1}) \\ &= \int_0^t P(X_{n+1} \geq t \mid X_n = u)\, dF_n(u) \\ &= \int_0^t \frac{\overline{G}(t)}{\overline{G}(u)}\, dF_n(u). &(2.5) \end{aligned}$$

Define $a_0(t) = 1$ and

$$a_n(t) = \int_0^t \frac{dF_n(u)}{\overline{G}(u)} \quad \text{for } n \geq 1.$$

Then $P[N(t) = n] = a_n(t)\overline{G}(t)$ for $n \geq 0$. Differentiating (2.4) and (2.5) with respect to t, we obtain

$$-dF_{n+1}(t) + dF_n(t) = \overline{G}(t)\frac{dF_n(t)}{\overline{G}(t)} - a_n(t)dG(t).$$

Thus $a_{n+1}(t) = \int_0^t \frac{dF_{n+1}(u)}{\overline{G}(u)}$

$$= \int_0^t \frac{a_n(u)dG(u)}{\overline{G}(u)} .$$

Let $A(t, q) = \sum_{n=0}^{\infty} q^n a_n(t)$. Then

$$A(t, q) = a_0(t) + q \sum_{n=0}^{\infty} q^n a_{n+1}(t)$$

$$= 1 + q \int_0^t A(u, q)\frac{dG(u)}{\overline{G}(u)} .$$

Differentiating with respect to t, this yields

$$A'(t, q) = qA(t, q)\frac{dG(t)}{\overline{G}(t)} .$$

Since $A(0, q) = 1$, this has a unique solution given by

$$A(t, q) = \overline{G}^{-q}(t).$$

Hence $\phi(t,\ q) = \sum_{n=0}^{\infty} q^n P[N(t) = n]$

$$= \sum_{n=0}^{\infty} q^n \bar{G}(t) a_n(t)$$

$$= \bar{G}(t) A(t,\ q)$$

$$= \bar{G}^p(t). \qquad \square$$

Thus in the FMR model, the expected number of failures before t is given by

$$E(N(t)) = \frac{\partial}{\partial q}\phi(t,\ q)\Big|_{q=1} = -\ln\bar{G}(t). \tag{2.6}$$

Using (2.2) and (2.6), we can compare the expected number of failures before t in the RAR model and FMR model. The expected number of failures before t is smaller in the RAR model (for large t) if and only if

$$\bar{G}(t) \le e^{-t/u}. \tag{2.7}$$

Let G be an IFRA distribution with mean μ. Then $-\frac{1}{t}\log\bar{G}(t)$ is increasing and cannot lie always below $1/\mu$ or always above $1/\mu$, because this would contradict the assumption that the mean of G is μ. Thus (2.7) is true for all large t and the expected number of failures before t is smaller in the RAR model than in the FMR model if G is IFRA. Similarly, if G is DFRA then the expected number of failures before t is smaller in the FMR model than in the RAR model, for all large t.

We can add a slight element of generality to the FMR model by assuming that the first unit has life distribution F and that G-minimal repairs are performed at all failures. We will call this the Extended Forever Minimal Repair (EFMR) model. The probability generating function of $N^*(t)$, the number of failures before t in this model is obtained below.

<u>Corollary 2.2</u>. The probability generating function of $N^*(t)$ is given by

$$\phi^*(t,\ q) = \overline{F}(t) + q\int_0^t \frac{\overline{G}^p(t)}{\overline{G}^p(s)}\, dF(s).$$

<u>Proof</u>.

$$\begin{aligned} \text{Since } \phi^*(t,\ q) &= \sum_{n=0}^{\infty} P[N^*(t) = n]q^n \\ &= \overline{F}(t) + \sum_{n=1}^{\infty} q^n \int_0^t P[N(t-s) = n-1]\, dF(s) \\ &= \overline{F}(t) + q\int_0^t \phi(t-s,\ q)\, dF(s), \end{aligned}$$

where $\phi(u,\ q)$ and $N(u)$ are as given in Theorem 2.1 except that the first unit has life distribution G with age s, and when a failure occurs at time w, the failed unit is replaced by a unit with life distribution G and age $w+s$. From (2.3), we have

$$\phi^*(t,q) = \overline{F}(t) + q\int_0^t \frac{\overline{G}^p(t)}{\overline{G}^p(s)}\, dF(s). \qquad \square$$

If F and G have proportional hazard rates, i.e. $\overline{G}(t) = \overline{F}^{\gamma}(t)$ for some $\gamma > 0$, then the above probability generating function becomes

$$\phi^*(t,\ q) = \left(1 - \frac{q}{1-p\gamma}\right)\overline{F}(t) + \frac{q}{1-p\gamma}\ \overline{F}^{\gamma p}(t).$$

Though this is a distribution function in t for all $\gamma > 0$, it is a convex combination of $\overline{F}$ and $\overline{F}^{\gamma p}$ only for $0 < \gamma \le 1$.

The stochastic processes $\{N(t),\ t \ge 0\}$ and $\{N^*(t) \ge 0\}$ are interesting processes in their own right. Theorem 2.1 shows that the marginal distribution of $N(t)$ is Poisson with parameter $-\ln\overline{G}(t)$. The number of failures before time t, $N(t)$, in the FMR model is exactly the number of record values before time t in a

sequence of i.i.d. random variables with common distribution G. More general results about N(t) can be found in the literature for record values. For example, Shorrock (1972) has shown that $\{N(t),\ t \ge 0\}$ is a non-homogeneous Poisson process. Our proof of Theorem 2.1 can be modified to establish the same result.

We can interpret the probability generating functions obtained in the FMR and EFMR models as the distribution function of the waiting time between two perfect repairs in another maintenance model due to Brown and Proschan. In the Brown and Proschan model, the system starts out with a unit with life distribution F. At each failure, a coin with probability p for heads is tossed independent of previous history. If it is a head, a perfect repair is performed and the failed unit is replaced by a unit whose life distribution is F. If the coin turns up tails, the unit is replaced by a unit with life distribution G and age t* where t* is the time from the beginning or the previous perfect repair, whichever came last. In short, when the coin turns up tails, a G-minimal repair is performed. We distinguish two models, the Brown-Proschan (BP) model in which $F = G$ and the Extended Brown-Proschan (EBP) model in which F and G are not equal. In both models, the epochs of perfect repair form regeneration points and the process starts over again. Thus it would be interesting to obtain the distribution of the waiting time for the first perfect repair or the time between two successive perfect repairs. Denote this waiting time by U in the BP model and by U* in the EBP model.

<u>Theorem 2.3</u>. (i) $P(U > t) = \phi(t,\ q) = \bar{G}^p(t)$.

(ii) $P(U^* > t) = \phi^*(t,\ q) = \bar{F}(t) + q\int_0^t \frac{\bar{G}^p(t)}{\bar{G}^p(s)}\, dF(s)$.

Proof.

We prove only (i). The proof of (ii) is similar.

$$\text{Since } P(U > t) = \sum_{n=0}^{\infty} P(\text{exactly } n \text{ failures occurred before } t \text{ and a G-minimal repair is performed at each failure})$$

$$= \sum_{n=0}^{\infty} q^n P[N(t) = n],$$

where $N(t)$ is the number of failures before t in the FMR model. From the definition of $\phi(t, q)$, we have

$$P(U > t) = \phi(t, q). \quad \square$$

This provides an alternative proof for Lemma 2.1 of Brown and Proschan (1983).

Brown and Proschan studied the ageing property of the distribution of U in terms of those of G. We give a result below that gives an upperbound for EU^* assuming that G is NBU or DMRL.

Theorem 2.4. Let G be NBU or DMRL. Then under the EBP model,

$$EU^* \leq \mu + \frac{q}{p} m \quad \text{for } 0 \leq p < 1$$

where $\mu = \int \overline{F}(t)dt$ and $m = \int \overline{G}(t)dt$.

Proof.

It follows from Brown and Proschan (1983) that $\overline{G}^p$ is NBUE and $\int \overline{G}^p(t)dt \leq (1/p) \int \overline{G}(t)dt$ for $0 < p \leq 1$. Thus

$$\frac{\int_u^\infty \bar{G}^p(t)\,dt}{\bar{G}^p(u)} \le \int_0^\infty \bar{G}^p(t)\,dt$$

$$\le m/p \quad \text{if } u \ge 0 \text{ and } 0 < p \le 1.$$

By Theorem 2.3, we have

$$\begin{aligned} EU^* &= \int \phi^*(t,\ q)\,dt \\ &= \mu + q\int_0^\infty [\bar{G}^p(t)\int_0^t \frac{dF(u)}{\bar{G}^p(u)}\,]dt \\ &= \mu + q\int_0^\infty [\,\int_u^\infty \bar{G}^p(t)dt/\bar{G}^p(u)]dF(u) \\ &\le \mu + q\int_0^\infty \bar{G}^p(t)\,dt \\ &\le \mu + \frac{q}{p}m. \qquad \square \end{aligned}$$

3. Maintenance models with two types of minimal repairs.

We extend the results of the previous section to models where two types of minimal repairs are available. This could correspond to the practical situation where repairmen with two types of training are called in to do the repair, or where repairs are performed with spares from deteriorating stockpiles from two different manufacturers, and so on.

We distinguish three maintenance models in which only minimal repairs are performed. In the One-shot Random Repair (OSR) model, a coin with probability λ for heads is tossed. If it turns up heads, a unit with life distribution G_1 is placed into service at time $t = 0$ and only G_1-minimal repairs are performed thereafter. If the coins turns up tails, the distribution G_2 is used in place of G_1 in the above.

In the Completely Random Repair (CRR) model, a coin is tossed as before

and depending on whether it turns up heads or tails, the unit put into service at time $t=0$ has life distribution G_1 or G_2. At each new failure, a coin with probability λ for heads is tossed independent of previous history. Depending on whether it turns up heads or tails, a G_1-minimal repair or a G_2-minimal repair is performed.

In the Alternating Repair (AR) model, a coin is tossed as in the previous models. If it turns up heads, then a unit with life distribution G_1 is placed into service at time $t=0$. From then on, we perform minimal repairs in alternate order, i.e. G_2-minimal repair, G_1-minimal repair, G_2-minimal repair, etc. If the coin turns up tails, the roles of G_1 and G_2 are reversed.

We first obtain expressions for the probability generating functions of the numbers of failures before t in the above three models.

Theorem 3.1. Let $N_1(t)$ be the number of failures before t in the OSR model. Its probability generating function is given by

$$\phi_1(t,\ q) = \lambda\overline{G}_1^p(t) + \overline{\lambda}\,\overline{G}_2^p(t), \text{ where } p = 1-q,$$

and its expected value is

$$EN_1(t) = -\,[\lambda\ell n\overline{G}_1(t) + \overline{\lambda}\ell n\overline{G}_2(t)]. \tag{3.1}$$

Proof.

This follows immediately from (2.3) and (2.6). □

For the CRR model, we first show that the probability generating function must satisfy certain differential equations. We assume that G_1 and G_2 are

absolutely continuous with respective densities g_1 and g_2.

<u>Theorem 3.2</u>. Let $N_2(t)$ be the number of failures before t in the CRR model. Its probability generating function is given by

$$\phi_2(t, q) = \lambda\bar{G}_1(t)A(t, q) + \bar{\lambda}\bar{G}_2(t)B(t, q)$$

where A(t, q), B(t, q) satisfy

$$(3.2) \qquad \bar{G}_1(t)\frac{\partial}{\partial t} A(t, q) = \bar{G}_2(t)\frac{\partial}{\partial t} B(t, q) = q[\lambda A(t, q)g_1(t) + \bar{\lambda}B(t, q)g_2(t)]$$

$$\text{for all } t > 0,$$

with initial conditions

$$A(0, q) = B(0, q) = 1,$$

and where $\bar{\lambda} = 1 - \lambda$.

<u>Proof</u>.

As in the proof of Theorem 2.1, we let X_n be the time of the n^{th} failure and let F_n be the distribution of X_n. Then

$$(3.3) \qquad P[N(t) = n] = \bar{F}_{n+1}(t) - \bar{F}_n(t)$$

$$= P(X_n < t \le X_{n+1})$$

$$= \int_0^t P(X_{n+1} \ge t | X_n = u)dF_n(u)$$

$$(3.4) \qquad = \lambda \int_0^t \frac{\bar{G}_1(t)}{\bar{G}_1(u)} dF_n(u) + \bar{\lambda}\int_0^t \frac{\bar{G}_2(t)}{\bar{G}_2(u)} dF_n(u)$$

$$= \lambda a_n(t)\bar{G}_1(t) + \bar{\lambda}b_n(t)\bar{G}_2(t)$$

where $a_n(t) = \int_0^t \frac{dF_n(u)}{\overline{G}_1(u)}$,

$$b_n(t) = \int_0^t \frac{dF_n(u)}{\overline{G}_2(u)} \quad \text{for } n \geq 1,$$

and $a_0(t) = b_0(t) = 1$.

Differentiating (3.3) and (3.4) with respect to t, we obtain the identity

$$- dF_{n+1}(t) + dF_n(t)$$

$$= \lambda\overline{G}_1(t)\frac{dF_n(t)}{\overline{G}_1(t)} - \lambda a_n(t)g_1(t) + \overline{\lambda}\,\overline{G}_2(t)\frac{dF_n(t)}{\overline{G}_2(t)} - \overline{\lambda}b_n(t)g_2(t)$$

$$= dF_n(t) - \lambda a_n(t)g_1(t) - \overline{\lambda}b_n(t)g_2(t).$$

This implies that for $n \geq 1$,

$$a_{n+1}(t) = \int_0^t \frac{dF_{n+1}(u)}{\overline{G}_1(u)}$$

(3.5)

$$= \lambda \int_0^t \frac{a_n(u)dG_1(u)}{\overline{G}_1(u)} + \overline{\lambda} \int_0^t \frac{b_n(u)dG_2(u)}{\overline{G}_1(u)},$$

and

$$b_{n+1}(t) = \lambda \int_0^t \frac{a_n(u)dG_1(u)}{\overline{G}_2(u)} + \overline{\lambda} \int_0^t \frac{b_n(u)dG_2(u)}{\overline{G}_2(u)}.$$

Let $A(t, q) = \sum_{n=0}^{\infty} q^n a_n(t)$ and $B(t,q) = \sum_{n=0}^{\infty} q^n b_n(t)$.

Then
$$\phi(t, q) = \sum_{n=0}^{\infty} P[N(t) = n]q^n$$

$$= \lambda\overline{G}_1(t)A(t, q) + \overline{\lambda}\,\overline{G}_2(t)B(t, q).$$

Note that $a_n(0) = b_n(0) = 0$ for $n \geq 1$, so that

$$A(0, q) = B(0, q) = 1.$$

Also

$$A(t, q) = 1 + q \sum_{n=0}^{\infty} q^n a_{n+1}(t)$$

$$= 1 + q \sum_{n=0}^{\infty} q^n \int_0^t \frac{dF_{n+1}(u)}{\overline{G}_1(u)}$$

$$= 1 + q \sum_{n=0}^{\infty} q^n \left[\lambda \int_0^t \frac{a_n(u)\,dG_1(u)}{\overline{G}_1(u)} + \overline{\lambda} \int_0^t \frac{b_n(u)\,dG_2(u)}{\overline{G}_1(u)}\right],$$

by (3.5).

Thus $\overline{G}_1(t)\frac{\partial}{\partial t} A(t, q) = q[\lambda A(t, q)g_1(t) + \overline{\lambda}\, B(t, q)g_2(t)]$. Similarly we have

$$\overline{G}_2(t)\frac{\partial}{\partial t} B(t, q) = q[\lambda A(t, q)g_1(t) + \overline{\lambda}\, B(t, q)g_2(t)]. \qquad \square$$

We are not able to obtain solutions of the linear homogeneous partial differential equations given in (3.2) for arbitrary G_1 and G_2. If G_1 and G_2 have proportional hazard rates, i.e.,

$$\overline{G}_1(t) = \overline{G}(t) \text{ and } \overline{G}_2(t) = \overline{G}^{\theta}(t) \quad \text{for some } 0 < \theta \leq 1, \tag{3.6}$$

then it easy to solve for $A(t, q)$ and $B(t, q)$. The solutions are

$$A(t, q) = \beta\overline{G}^{\alpha_1}(t) + \overline{\beta}\,\overline{G}^{\alpha_2}(t)$$

and

$$B(t, q) = \frac{\alpha_1\beta}{\alpha_1 - \theta + 1}\, \overline{G}^{\alpha_1 - \theta + 1}(t) + \frac{\alpha_2\overline{\beta}}{\alpha_2 - \theta + 1}\, \overline{G}^{\alpha_2 - \theta + 1}(t)$$

where α_1, α_2 are the roots of the equation

$$(\alpha + q\lambda)(\alpha - \theta + 1) + \alpha q\bar{\lambda}\theta = 0$$

and $\bar{\beta} = 1 - \beta$ is chosen such that

$$\frac{\alpha_1\beta}{\alpha_1 - \theta + 1} + \frac{\alpha_2\bar{\beta}}{\alpha_2 - \theta + 1} = 1.$$

Note that α_1 and α_2 are real because

$$[q\lambda + (1 - \theta) + q\bar{\lambda}\theta]^2 - 4q\lambda(1 - \theta)$$

$$= [q\lambda - (1 - \theta)]^2 + (q\bar{\lambda}\theta)^2 + 2q\bar{\lambda}\theta[q\alpha + (1 - \theta)] > 0.$$

Thus the probability generating function of $N_2(t)$ is

$$\phi_2(t, q) = [\lambda\beta + \bar{\lambda}\frac{\alpha_1\beta}{\alpha_1 - \theta + 1}]\,\bar{G}^{\alpha_1+1}(t)$$

$$+ [\lambda\bar{\beta} + \bar{\lambda}\frac{\alpha_2\bar{\beta}}{\alpha_2 - \theta + 1}]\,\bar{G}^{\alpha_2+1}(t).$$

We find that the expected number of failures before t, under the assumption (3.6), is given by

$$\text{(3.7)} \qquad EN_2(t) = -\frac{\theta}{\bar{\lambda} + \lambda\theta}\,\ell n\bar{G}(t) + \frac{\lambda\bar{\lambda}(1 - \theta)^2}{(\bar{\lambda} + \lambda\theta)^2}\,[1 - \bar{G}^{\,\bar{\lambda}+\lambda\theta}(t)].$$

We just noted that we were unable to obtain the probability generating function of $N_2(t)$ for arbitrary G_1 and G_2. However we are able to obtain $EN_2(t)$ for arbitrary G_1 and G_2 in Theorem 3.3 below. This is possible because the identity (3.12) (see below) allows us to reduce the differential equations for $A_1(t)$ and $B_1(t)$ which determine $EN_2(t)$ (see (3.15) below) and solve for them. Let r_1, r_2 denote the respective failure rates functions of G_1 and G_2.

Theorem 3.3. In the CRR model, we have

$$(3.8)\quad EN_2(t) = -[\lambda \ln \bar{G}_1(t) + \bar{\lambda} \ln \bar{G}_2(t)]$$
$$-\lambda\bar{\lambda}\int_0^t \bar{G}_1^{\bar{\lambda}}(u)\bar{G}_2^{\lambda}(u)[r_1(u) - r_2(u)]\left\{\int_0^u \frac{r_1(v) - r_2(v)}{\bar{G}_1^{\bar{\lambda}}(v)\bar{G}_2^{\lambda}(v)}dv\right\}du.$$

Proof.

By Theorem 3.2, we have

$$(3.9)\quad \phi_2(t, q) = \lambda\bar{G}_1(t)A(t, q) + \bar{\lambda}\bar{G}_2(t)B(t, q)$$

where

$$(3.10)\quad A(t, q) - 1 = q\lambda\int_0^t \frac{A(u, q)dG_1(u)}{\bar{G}_1(u)} + q\bar{\lambda}\int_0^t \frac{B(u, q)dG_2(u)}{\bar{G}_1(u)}$$

and

$$B(t, q) - 1 = q\lambda\int_0^t \frac{A(u, q)dG_1(u)}{\bar{G}_2(u)}$$
$$+ q\bar{\lambda}\int_0^t \frac{B(u, q)dG_2(u)}{\bar{G}_2(u)} .$$

Let $A_1(t) = A(t, 1)$ and $B_1(t) = B(t, 1)$. Then (3.10) implies that

$$(3.11)\quad \begin{aligned}\bar{G}_1(t)A_1'(t) &= \bar{G}_2(t)B_1'(t)\\ &= \lambda A_1(t)g_1(t) + \bar{\lambda}B_1(t)g_2(t).\end{aligned}$$

Putting $q = 1$ in (3.9), we have

$$(3.12)\quad \lambda\bar{G}_1(t)A_1(t) + \bar{\lambda}\bar{G}_2(t)B_1(t) = 1.$$

Solving for $A_1(t)$ and $B_1(t)$ from (3.9) and (3.10), we obtain

$$A_1'(t)\bar{G}_1(t)\bar{G}_2(t) = \lambda A_1(t)[\bar{G}_2(t)g_1(t) - \bar{G}_1(t)g_2(t)] + g_2(t),$$

which implies that

$$\frac{d}{dt}[A_1(t)\bar{G}_1^{\lambda}(t)\bar{G}_2^{-\lambda}(t)] = \frac{\bar{G}_1^{\lambda}(t)}{\bar{G}_2^{\lambda}(t)}\{A_1'(t) + \lambda[r_2(t) - r_1(t)]A_1(t)\}$$

$$= \frac{\bar{G}_1^{\lambda}(t)}{\bar{G}_2^{\lambda}(t)} \frac{g_2(t)}{\bar{G}_1(t)\bar{G}_2(t)}.$$

This yields the solution

$$A_1(t) = \frac{\bar{G}_2^{\lambda}(t)}{\bar{G}_1^{\lambda}(t)}\left[1 + \int_0^t \frac{dG_2(u)}{\bar{G}_1^{\bar{\lambda}}(u)\bar{G}_2^{1+\lambda}(u)}\right]$$

Similarly

$$B_1(t) = \frac{\bar{G}_1^{\bar{\lambda}}(t)}{\bar{G}_2^{\bar{\lambda}}(t)}\left[1 + \int_0^t \frac{dG_1(u)}{\bar{G}_1^{1+\bar{\lambda}}(u)\bar{G}_2^{\lambda}(u)}\right].$$

To solve for $EN_2(t)$, we let $E(t) = \frac{\partial}{\partial q} A(t, q)\big|_{q=1}$ and $F(t) = \frac{\partial}{\partial q} B(t, q)\big|_{q=1}$. By differentiating (3.10) with respect to q and putting $q = 1$, we obtain

$$E(t) = \lambda \int_0^t \frac{A_1(u)dG_1(u)}{\bar{G}_1(u)} + \bar{\lambda}\int_0^t \frac{B_1(u)dG_2(u)}{\bar{G}_1(u)}$$

$$+ \lambda \int_0^t \frac{E(u)dG_1(u)}{\bar{G}_1(u)} + \bar{\lambda}\int_0^t \frac{F(u)dG_2(u)}{\bar{G}_1(u)}$$

$$= A_1(t) - 1 + \lambda \int_0^t \frac{E(u)dG_1(u)}{\bar{G}_1(u)} + \bar{\lambda}\int_0^t \frac{F(u)dG_2(u)}{\bar{G}_1(u)}.$$

The last equality follows from (3.10). Thus,

$$E'(t)\overline{G}_1(t) = A_1'(t)\overline{G}_1(t) + \lambda E(t)g_1(t) + \overline{\lambda}F(t)g_2(t). \tag{3.13}$$

Similarly,

$$F'(t)\overline{G}_2(t) = B_1'(t)\overline{G}_2(t) + \lambda E(t)g_1(t) + \overline{\lambda}F(t)g_2(t). \tag{3.14}$$

Since
$$\begin{aligned}\frac{d}{dt}[EN_2(t)] &= \frac{d}{dt}\left[\frac{\partial}{\partial q}\phi_2(t, q)\Big|_{q=1}\right] \\ &= \frac{d}{dt}[\lambda\overline{G}_1(t)E(t) + \overline{\lambda}\,\overline{G}_2(t)F(t)] && \text{by (3.9)} \\ &= \lambda E'(t)\overline{G}_1(t) + \overline{\lambda}F'(t)\overline{G}_2(t) \\ &\quad - \lambda E(t)g_1(t) - \overline{\lambda}F(t)g_2(t) \\ &= \lambda A_1'(t)\overline{G}_1(t) + \overline{\lambda}B_1'(t)\overline{G}_2(t) && \text{by (3.13) and (3.14)} \\ &= \lambda A_1(t)g_1(t) + \overline{\lambda}B_1(t)g_2(t) && \text{by (3.11)}\end{aligned}$$

we have

$$EN_2(t) = \lambda\int_0^t A_1(u)dG_1(u) + \overline{\lambda}\int_0^t B_1(u)dG_2(u). \tag{3.15}$$

Substituting the solution of $A_1(t)$ and $B_1(t)$, we obtain

$$\begin{aligned}EN_2(t) = {} & \lambda\int_0^t \frac{\overline{G}_2^{\lambda}(u)}{\overline{G}_1^{\lambda}(u)}\,dG_1(u) + \overline{\lambda}\int_0^t \frac{\overline{G}_1^{\overline{\lambda}}(u)}{\overline{G}_2^{\overline{\lambda}}(u)}\,dG_2(u) \\ & + \lambda\int_0^t\int_0^u \frac{\overline{G}_2^{\lambda}(u)}{\overline{G}_1^{\lambda}(u)}\,\frac{1}{\overline{G}_1^{\overline{\lambda}}(v)\overline{G}_2^{1+\lambda}(v)}\,dG_2(v)dG_1(u) \\ & + \overline{\lambda}\int_0^t\int_0^u \frac{\overline{G}_1^{\overline{\lambda}}(u)}{\overline{G}_2^{\overline{\lambda}}(u)}\,\frac{1}{\overline{G}_1^{1+\overline{\lambda}}(v)\overline{G}_2^{\lambda}(v)}\,dG_1(v)dG_2(u).\end{aligned}$$

Integrating by parts, we have

$$EN_2(t) = -\ell n\overline{G}_1(t) - \overline{\lambda}\int_0^t \left[\frac{\overline{G}_2(u)}{\overline{G}_1(u)}\right]^{\lambda} dG_1(u)$$

$$(3.16) \qquad + \overline{\lambda}\int_0^t \left[\frac{\overline{G}_1(u)}{\overline{G}_2(u)}\right]^{\overline{\lambda}} dG_2(u)$$

$$+ \overline{\lambda}\int_0^t \left[\frac{\overline{G}_1^{\overline{\lambda}}(u)}{\overline{G}_2^{\overline{\lambda}}(u)}\, g_2(u) - \frac{\overline{G}_2^{\lambda}(u)}{\overline{G}_1^{\lambda}(u)}\, g_1(u)\right]\left[\int_0^u \frac{dG_1(v)}{\overline{G}_1^{1+\overline{\lambda}}(v)\overline{G}_2^{\lambda}(v)}\right] du.$$

Interchanging G_1 with G_2 and λ with $\overline{\lambda}$ in (3.16), we obtain the following alternative expression for $EN_2(t)$:

$$EN_2(t) = -\ell n\overline{G}_2(t) - \lambda\int_0^t \left[\frac{\overline{G}_1(u)}{\overline{G}_2(u)}\right]^{\overline{\lambda}} dG_2(u)$$

$$(3.17) \qquad + \lambda\int_0^t \left[\frac{\overline{G}_2(u)}{\overline{G}_1(u)}\right]^{\lambda} dG_1(u)$$

$$+ \lambda\int_0^t \left[\frac{\overline{G}_2^{\lambda}(u)}{\overline{G}_1^{\lambda}(u)}\, g_1(u) - \frac{\overline{G}_1^{\overline{\lambda}}(u)}{\overline{G}_2^{\overline{\lambda}}(u)}\, g_2(u)\right]\left[\int_0^u \frac{dG_2(v)}{\overline{G}_1^{\overline{\lambda}}(v)\overline{G}_2^{1+\lambda}(v)}\right] du.$$

The expression for $EN_2(t)$ in (3.8) is obtained from (3.16) and (3.17) by appropriate averaging. □

We next present analogous results for the AR model. Let $N_3(t)$ be the number of failures before t in the AR model.

Lemma 3.4. Consider the conditional distribution of $N_3(t)$ given that a head turns up on the first toss, which is the same as saying that the first unit has life distribution G_1. Its probability generating function is

$$\bar{G}_1(t)A(t, q) + \bar{G}_2(t)B(t, q),$$

where $A(t, q)$ and $B(t, q)$ satisfy

$$\bar{G}_1(t)\frac{\partial}{\partial t} A(t, q) = qB(t, q)g_2(t),$$

$$\bar{G}_2(t)\frac{\partial}{\partial t} B(t, q) = qA(t, q)g_1(t),$$

with initial conditions

$$A(0, q) = 1, \; B(0, q) = 0.$$

The expected value of $N_3(t)$ given that the first unit has life distribution G_1 is

$$-\ell n\bar{G}_2(t) + \int_0^t \bar{G}_1(u)\bar{G}_2(u)[r_1(u) - r_2(u)]\left\{1 + \int_0^u \frac{dG_2(v)}{\bar{G}_1(v)\bar{G}_2^2(v)}\right\} du.$$

Proof.

The proof is omitted because it is similar to the proof of Theorems 3.2 and 3.3. □

The expected value of $N_3(t)$ follows immediately from Lemma 3.4. We state this result without giving the proof.

Theorem 3.5. In the AR model,

$$EN_3(t) = -[\lambda \ln \bar{G}_2(t) + \bar{\lambda} \ln \bar{G}_1(t)]$$

$$(3.18) \qquad - \int_0^t \bar{G}_1(u)\bar{G}_2(u)[r_1(u) - r_2(u)]\left\{-(\lambda - \bar{\lambda}) + \bar{\lambda}\int_0^u \frac{dG_1(v)}{\bar{G}_1^2(v)\bar{G}_2(v)} - \lambda \int_0^u \frac{dG_2(v)}{\bar{G}_1(v)\bar{G}_2^2(v)}\right\} du.$$

We may now compare the expected number of failures among the three models for the case where $\lambda = \bar{\lambda} = 1/2$. We further assume that the failure rate functions satisfy

$$r_1(t) \geq r_2(t) \quad \text{for all } t \geq 0.$$

Then it follows from (3.1), (3.8) and (3.18) that the expected number of failures before t is the largest in the OSR model. To compare the AR model and the CRR model, it is clear from (3.8) and (3.18) that the expected number of failures before t is smaller in the AR model than in the CRR model if and only if

$$1/2 \int_0^t \bar{G}_1^{1/2}(u)\bar{G}_2^{1/2}(u)[r_1(u) - r_2(u)]\left\{\int_0^u \frac{r_1(v) - r_2(v)}{\bar{G}_1^{1/2}(v)\bar{G}_2^{1/2}(v)}\, dv\right\} du$$

(3.19)

$$\leq \int_0^t \bar{G}_1(u)\bar{G}_2(u)[r_1(u) - r_2(u)]\left\{\int_0^u \frac{r_1(v) - r_2(v)}{\bar{G}_1(v)\bar{G}_2(v)}\, dv\right\} du.$$

We were not able to prove (3.19) for all $t \geq 0$. However, for small values of t, namely when $\bar{G}_1(t)\bar{G}_2(t) \geq 1/4$, it is straight forward to show that (3.19) is true for arbitrary G_1 and G_2. Therefore, if the process is to be run for a short time, AR model is preferable to the CRR model. Under the proportional hazard

assumption, namely (3.6), the expected number of failures before t is smaller in the AR model than in the CRR model for all t, since (3.19) can be reduced to the inequality

$$1/2[1 - \bar{G}^{1+\theta}(t)] \le 1 - \bar{G}^{(1+\theta)/2}(t),$$

which holds for all $t \ge 0$.

The BP model of Section 2 can be extended to include perfect repair and two kinds of minimal repairs. Once again epoch of perfect repairs form regeneration points. The tail of the distribution of the waiting time between perfect repairs is again the probability generating function of the number of failures in the same repair model when perfect repairs are excluded.

Repair models with k minimal repairs where $k \ge 3$ can be obtained by straight forward extensions. Differential equations for the various quantities involved similar to those found in Theorem 3.2 are easily obtained. It turns out that these linear homogeneous differential equations are not easy to solve even in the proportional hazard case.

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